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# On the Structure of Fixed Point Sets of Compact Maps in $B_0$ Spaces with Applications to Integral and Differential Equations in Unbounded Domain

KRZYSZTOF CZARNOWSKI AND TADEUSZ PRUSZKO

*Institute of Mathematics, University of Gdansk, ul. Wita Stwosza 57, 80-952 Gdansk, Poland*

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In this paper we present certain applications of the topological degree theory in locally convex linear topological spaces. As a result we obtain two generalizations of the Krasnosielski–Perov–Rabinovitz theorem on the characterization of sets of solutions of functional equations, known for the Banach space case, onto the case of a locally convex space. On the other hand we make use of such consequences of the degree theory as the Borsuk theorem and Browder domain invariance theorem in order to reach a certain generalization of the Lasota–Opial covering theorem and a new theorem on the characterization of sets of solutions of functional equations. At the end we apply the abstract topological theorems to study the structure of sets of solutions of a certain integral equation and the Darboux problem for a hyperbolic equation in an unbounded domain. A general assumption is that the kernel of the integral equation and the right-hand side of the hyperbolic equation satisfy the Caratheodory conditions.

## 1. PRELIMINARIES

Let $X$ denote a metric space. $X$ is a compact absolute retract ($X \in \mathscr{A}\mathscr{R}$) if $X$ is compact and, for every homeomorphism $f: X \to f(X)$ into a metric space $Y$, $f(X)$ is a retract of $Y$. It follows from the Dugundji extension theorem that every compact and convex subset of a metrizable locally convex space is a compact absolute retract. $A \subseteq X$ is an $\mathscr{R}_\delta$-set in the space $X$ if $A = \bigcap_{n=1}^{\infty} A_n$ for a decreasing sequence $\{A_n\}$ of compact absolute retracts contained in $X$.

(1.1) THEOREM (Aronszajn [1]). *Suppose that $A$, $A_n$ ($n \in \mathbb{N}$) are compact subsets of the space $X$. If $A \subseteq A_n$ and $A_n \in \mathscr{A}\mathscr{R}$ for every $n \in \mathbb{N}$, and the Hausdorff distance $d(A_n, A) \to 0$ when $n \to \infty$, then $A$ is an $\mathscr{R}_\delta$-set in $X$.*



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In what follows $E$ denotes a real Frechet space (or in other words a $B_0$ space) and $U \subseteq E$ an open subset of $E$. The topology of $E$ is induced by a sequence of seminorms $\{q_n: E \to \mathbb{R}_+ \mid n \in \mathbb{N}\}$ which can be assumed to be nondecreasing: $q_n(x) \leqslant q_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in E$. Recall that the topology of $E$ coincides with that of a complete metric space $\langle E, d \rangle$, where

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{q_n(x-y)}{1+q_n(x-y)},$$

and $d(x_n, x) \to 0$ if and only if $q_n(x_n - x) \to 0$ for any sequence $\{x_n\} \subseteq E$. If

$$U_n = \{x \in E \mid q_n(x) < 1\} \qquad (n \in \mathbb{N})$$

then $\{\varepsilon U_n \mid n \in \mathbb{N}, \varepsilon > 0\}$ is a basis of convex and symmetric neighbourhoods of zero in the space $E$. Let us also note that

(1.2) if $A, A_n \subseteq E$ $(n \in \mathbb{N})$ are closed sets such that $A \subseteq A_n$ for every $n \in \mathbb{N}$ and $d(A_n, A) \to 0$ then for every neighbourhood of zero $V$ there exists $N \in \mathbb{N}$ such that $A_n \subseteq A + V$ for all $n > N$.

A continuous map $F: X \to E$ is compact if $F(X)$ is a relatively compact subset of $E$. If $F: \bar{U} \to E$ is a compact map then the map $f: \bar{U} \to E$, $f(x) = x - F(x)$ $(\equiv (I-F)(x))$ is a compact vector field in $U$. A topological degree $\deg(f, U, y) \in \mathbb{Z}$ is defined for a compact vector field $f$ in $U$ and $y \notin f(\partial U)$ such that (see [6])

(1.3) $\deg(f, U, y) = 0$ for $y \notin f(\bar{U})$,

(1.4) if $H: [0, 1] \times \bar{U} \to E$ is a compact homotopy connecting compact maps $F_0 - y_0$, $F_1 - y_1: \bar{U} \to E$ such that

$$\operatorname{Fix} H \equiv \{x \in \bar{U} \mid \exists_{t \in [0,1]}\, x = H(t, x)\} \subseteq U,$$

then $\deg(I - F_0, U, y_0) = \deg(I - F_1, U, y_1)$,

(1.5) if $y_1$ and $y_2$ lie in the same component of the set $E \setminus f(\partial U)$ then $\deg(f, U, y_1) = \deg(f, U, y_2)$.

Moreover the following domain invariance theorem holds.

(1.6) THEOREM (Nagumo [6]). *If $f: \bar{U} \to E$ is an injective compact vector field then $f(U)$ is an open set in $E$ and for every $y \in f(U)$* $\deg(f, U, y) = \pm 1$.

A map $H: X \to n(E)$, where $n(E)$ is the family of all nonempty subsets of $E$, is upper semicontinuous (u.s.c.) if the graph of $H$ is closed in $X \times E$. A (u.s.c.) map $H: X \to n(E)$ is compact if the set $H(X)$ is relatively compact in $E$.

## 2. RESULTS

(2.1) DEFINITION. Let $f: \bar{U} \to E$ be a continuous map and $\{\varepsilon_n\}$ a sequence of positive reals tending to zero. A sequence of continuous maps $\{f_n: \bar{U} \to E \mid n \in \mathbb{N}\}$ is an $\{\varepsilon_n\}$-approximation of the map $f$ if $q_n(f_n(x) - f(x)) \leqslant \varepsilon_n$ for all $n \in \mathbb{N}$ and $x \in \bar{U}$.

The following theorem is a generalization of the Krasnosielski–Perov–Rabinovitz theorem known for compact maps in Banach spaces [5].

(2.2) THEOREM. *Suppose that $F: \bar{U} \to E$ is a compact map satisfying the following conditions*:

(2.2.1) $0 \notin (I-F)(\partial U)$ *and* $\deg(I-F, U, 0) \neq 0$,

(2.2.2) *there exists a compact $\{\varepsilon_n\}$-approximation $\{F_n: \bar{U} \to E \mid n \in \mathbb{N}\}$ of $F$ such that the equation $x - F_n(x) = y$ has at most one solution for every $n \in \mathbb{N}$ and $y \in \varepsilon_n \bar{U}_n$.*
*Then the set of fixed points Fix $F$ is an $\mathscr{R}_\delta$-set in $E$.*

(2.3) THEOREM. *If $F: E \to E$ is a compact map and there exists a compact $\{\varepsilon_n\}$-approximation $\{F_n: E \to E \mid n \in \mathbb{N}\}$ of $F$ such that the equation $x - F_n(x) = y$ has at most one solution for every $n \in \mathbb{N}$ and $y \in \varepsilon_n \bar{U}_n$, then Fix $F$ is an $\mathscr{R}_\delta$-set in $E$.*

(2.4) DEFINITION. A continuous map $F: E \to E$ satisfies the Lasota–Opial condition iff there exist a convex and symmetric neighbourhood of zero $\mathcal{O}$ and a compact map $H: \bar{\mathcal{O}} \to \varkappa(E)$ such that

(2.4.1) $x \in H(x)$ implies $x = 0$ for every $x \in \bar{\mathcal{O}}$,

(2.4.2) $F(x) - F(y) \in H(x-y)$ for every $x, y \in E$ such that $x - y \in \bar{\mathcal{O}}$.

The next theorem is the Lasota–Opial theorem [4] generalized from the case of a Banach space to the case of a Frechet space.

(2.5) THEOREM. *If $F: E \to E$ satisfies the Lasota–Opial condition than the map $(I-F): E \to E$ is a homeomorphism.*

(2.6) THEOREM. *If $F: E \to E$ is a compact map and there exists an $\{\varepsilon_n\}$-approximation $\{F_n: E \to E \mid n \in \mathbb{N}\}$ of $F$ such that $F_n$ satisfies the Lasota–Opial condition for every $n \in \mathbb{N}$, then Fix $F$ is an $\mathscr{R}_\delta$-set in $E$.*

Let us accept the following denotations: $\mathbb{R}_+ = [0, +\infty)$, $\Delta = \mathbb{R}^\lambda_+$ $(\lambda = 1, 2, 3, \ldots)$, $\Gamma = \mathbb{R}^\mu_+$ $(\mu = 0, 1, 2, \ldots)$ and for $a > 0$: $\Delta^a = [0, a]^\lambda$, $\Gamma^a = [0, a]^\mu$. In applications we consider maps $K: \Delta \times \Gamma \times \mathbb{R}^\nu \to \mathbb{R}^\nu$ $(\nu \in \mathbb{N})$ satisfying the following assumptions.

(2.7) *Assumptions.*

(2.7.1) the map $K(s, \cdot, \cdot): \Gamma \times \mathbb{R}^\nu \to \mathbb{R}^\nu$ is continuous for every $s \in \Delta$,

(2.7.2) the map $K(\cdot, t, x): \Delta \to \mathbb{R}^\nu$ is Lebesque measurable for every $(t, x) \in \Gamma \times \mathbb{R}^\nu$,

(2.7.3) there exist locally integrable functions $p, c: \Delta \to \mathbb{R}_+$ such that $|K(s, t, x)| \leqslant p(s)|x| + c(s)$ for all $(s, t, x) \in \Delta \times \Gamma \times \mathbb{R}^\nu$.

We say that a map $w: \Delta \to \mathbb{R}^\nu$ given by the formula $w(s) = (w_1(s), ..., w_\nu(s))$ is absolutely continuous if $w_i: \Delta^a \to \mathbb{R}$ is an absolutely continuous function for every $a > 0$ and $i = 1, ..., \nu$.

(2.8) THEOREM. *Suppose that a map $K: \Delta \times \mathbb{R}^\nu \to \mathbb{R}^\nu$ satisfies the assumptions* (2.7), *where $\lambda = 2$ and $\mu = 0$, and $g, h: \mathbb{R}_+ \to \mathbb{R}^\nu$ are absolutely continuous maps such that $g(0) = h(0)$. Then the set of all solutions of the Darboux problem*

$$\begin{aligned} &u_{xy}(x, y) = K(x, y, u(x, y)), \qquad \textit{for} \quad \textit{a.a.}\ (x, y) \in \Delta, \\ &u(0, y) = g(y),\ u(x, 0) = h(x), \qquad \textit{for} \quad x, y \in \mathbb{R}_+ \\ &u: \Delta \to \mathbb{R}^\nu \ \textit{is an absolutely continuous map}, \end{aligned} \tag{D}$$

*is an $\mathscr{R}_\delta$-set in the space of continuous maps $\langle \mathscr{C}(\Delta, \mathbb{R}^\nu), \{q_n \mid n \in \mathbb{N}\} \rangle$, where $q_n(u) = \sup\{|u(x, y)| \mid (x, y) \in \Delta^n\}$.*

(2.9) THEOREM. *If a map $K: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\nu \to \mathbb{R}^\nu$ satisfies the assumptions* (2.7), *where $\lambda = \mu = 1$, then the set of all continuous solutions of the integral equation*

$$x(t) = \int_0^t K(s, t, x(s))\, ds \tag{I}$$

*is an $\mathscr{R}_\delta$-set in the space $\langle \mathscr{C}(\mathbb{R}_+, \mathbb{R}^\nu), \{q_n \mid n \in \mathbb{N}\} \rangle$, where $q_n(x) =$* $\sup\{|x(t)| \mid t \in [0, n]\}$.

## 3. AUXILIARY LEMMAS

(3.1) LEMMA. *Suppose that a continuous map $f: \bar{U} \to E$ satisfies the following conditions:*

(3.1.1) *every sequence $\{x_n\}$ such that $f(x_n) \to 0$ contains a convergent subsequence (Palais–Smale condition),* (3.1.2) *there exists an* $\{\varepsilon_n\}$*-approximation* $\{f_n : \bar{U} \to E \mid n \in \mathbb{N}\}$ *of* $f$ *such that the map*

$$\bar{f}_n : f_n^{-1}(\varepsilon_n \bar{U}_n) \to \varepsilon_n \bar{U}_n, \qquad \bar{f}_n(x) = f_n(x)$$

*is a homeomorphism for every* $n \in \mathbb{N}$. *Then* $f^{-1}(0)$ *is an* $\mathcal{R}_\delta$-set *in* $E$.

*Proof.* For every $n \in \mathbb{N}$ there exists, by (3.1.2), $x_n \in E$ such that $f_n(x_n) = 0$. We have $q_n(f(x_n)) = q_n(f(x_n) - f_n(x_n)) \leqslant \varepsilon_n$ and so $f(x_n) \to 0$. We can assume, passing if necessary to a suitable subsequence, that $x_n \to x$. By the continuity of $f$, $f(x) = 0$; hence $f^{-1}(0) \neq \varnothing$. Moreover the continuity of $f$ and (3.1.1) imply compactness of $f^{-1}(0)$.

Let us note that $f^{-1}(0) \subseteq f_n^{-1}(\varepsilon_n \bar{U}_n)$ for every $n \in \mathbb{N}$. We show that the sequence of sets $\{f_n^{-1}(\varepsilon_n \bar{U}_n)\}$ converges to $f^{-1}(0)$ in the sense of Hausdorff metric. Suppose the contrary. By virtue of (1.2) there exists a neighbourhood of zero $V$, an increasing sequence of positive integers $\{n_k\}$, and a sequence $\{x_{n_k}\} \subseteq E$ such that $x_{n_k} \in f_{n_k}^{-1}(\varepsilon_{n_k} \bar{U}_{n_k})$ and $x_{n_k} \notin f^{-1}(0) + V$ for every $k \in \mathbb{N}$. We have

$$q_{n_k}(f(x_{n_k})) \leqslant q_{n_k}(f(x_{n_k}) - f_{n_k}(x_{n_k})) + q_{n_k}((f_{n_k}(x_{n_k}))) \leqslant 2\varepsilon_{n_k}$$

for every $k \in \mathbb{N}$; hence $f(x_{n_k}) \to 0$ and we can assume that $x_{n_k} \to x$. By the continuity of $f$, $x \in f^{-1}(0)$, and so $x_{n_k} \notin x + V$ for every $k \in \mathbb{N}$, which is impossible since $x_{n_k} \to x$.

For all $n \in \mathbb{N}$ the inclusions

$$f^{-1}(0) \subseteq \bar{f}_n^{-1}(\overline{\operatorname{conv}} f_n(f^{-1}(0))) \subseteq f_n^{-1}(\varepsilon_n \bar{U}_n)$$

hold. For every $n \in \mathbb{N}$ the set $\overline{\operatorname{conv}} f_n(f^{-1}(0))$ is compact and convex; hence it is a compact abslute retract as well as its homeomorphic image $\bar{f}_n^{-1}(\overline{\operatorname{conv}} f_n(f^{-1}(0)))$. Moreover, the sequence of sets $\{\bar{f}_n^{-1}(\overline{\operatorname{conv}} f_n(f^{-1}(0)))\}$ converges, along with $\{f_n^{-1}(\varepsilon_n \bar{U}_n)\}$, to $f^{-1}(0)$ in the sense of Hausdorff metric. By virtue of the Aronszajn theorem (1.1), $f^{-1}(0)$ is an $\mathcal{R}_\delta$-set in $E$. ∎

In the proofs of Theorems (2.8) and (2.9) we use the following approximation lemma.

(3.2) LEMMA. *Let us assume that* $K : \Delta \times \Gamma \times \mathbb{R}^\nu \to \mathbb{R}^\nu$ *satisfies the assumptions* (2.7) *and*

(3.2.1) *there exists a locally integrable function* $\alpha : \Delta \to \mathbb{R}_+$ *such that* $\operatorname{supp} K \subseteq \Omega_\alpha$, *where* $\operatorname{supp} K = K^{-1}(\mathbb{R}^\nu \setminus \{0\})$ *and* $\Omega_\alpha = \{(s, t, x) \in \Delta \times \Gamma \times \mathbb{R}^\nu \mid |x| \leqslant \alpha(s)\}$.

*Then for every* $\varepsilon > 0$ *and* $a > 0$ *there exists a map* $\bar{K}: \Delta \times \Gamma \times \mathbb{R}^\nu \to \mathbb{R}^\nu$ *which satisfies* (2.7), (3.2.1), *and the following two conditions*:

(3.2.2) *there exists an integrable function* $\varphi: \Delta^a \to \mathbb{R}_+$ *such that* $|\bar{K}(s,t,x) - K(s,t,x)| \leqslant \varphi(s)$ *for all* $(s,t,x) \in \Delta^a \times \Gamma^a \times \mathbb{R}^\nu$ *and* $\int_{\Delta^a} \varphi(s)\, ds < \varepsilon$,

(3.2.3) *there exists a locally integrable function* $L: \Delta \to \mathbb{R}_+$ *such that* $|\bar{K}(s,t,x) - \bar{K}(s,t,x')| \leqslant L(s)|x - x'|$ *for all* $s \in \Delta$, $t \in \Gamma$, *and* $x, x' \in \mathbb{R}^\nu$.

*Proof.* For every $\delta > 0$ we define a map $K_\delta: \Delta \times \Gamma \times \mathbb{R}^\nu \to \mathbb{R}^\nu$,

$$K_\delta(s,t,x) = \int_{\mathbb{R}^\nu} K(s,t,x - \delta y)\, \psi(y)\, dy,$$

where $\psi: \mathbb{R}^\nu \to \mathbb{R}_+$ is a smooth test function: $\operatorname{supp} \psi = \bar{B}_\nu(1)$ $(\bar{B}_\nu(r) = \{x \in \mathbb{R}^\nu \mid |x| \leqslant r\})$, $\int_{\mathbb{R}^\nu} \psi(x)\, dx = 1$. The map $K_\delta$ satisfies (2.7), (3.2.1), and (3.2.3) (with some functions $p_\delta$, $c_\delta$, $\alpha_\delta$, and $L_\delta$) for every $\delta > 0$.

Let $\varepsilon > 0$ and $a > 0$ be arbitrarily chosen. We define a set

$$S_m = \{s \in \Delta^a \mid \forall_{t \in \Gamma^a} \forall_{x,x' \in \mathbb{R}^\nu}\ |x - x'| < 1/m \Rightarrow |K(s,t,x) - K(s,t,x')| \leqslant \varepsilon/2a^\lambda\}$$

for every $m \in \mathbb{N}$. Since the maps $K(\cdot, t, x)$ $((t,x) \in \Gamma \times \mathbb{R}^\nu)$ are measurable and the maps $K(s, \cdot, \cdot)$ $(s \in \Delta)$ are continuous, $S_m$ is measurable for $m \in \mathbb{N}$. Moreover $S_m \subseteq S_{m+1}$ for $m \in \mathbb{N}$ and, since the map $K(s, \cdot, \cdot)$ is uniformly continuous on $\Gamma^a \times \mathbb{R}^\nu$ for every $s \in \Delta^a$, $\bigcup_{m=1}^\infty S_m = \Delta^a$. Hence there exists $m' \in \mathbb{N}$ such that

$$\int_{Z_{m'}} (p(s)\, \alpha(s) + c(s))\, ds < \varepsilon/2,$$

where $Z_{m'} = \Delta^a \backslash S_{m'}$.

Let us choose $\delta = 1/m'$ and define $\bar{K}: \Delta \times \Gamma \times \mathbb{R}^\nu \to \mathbb{R}^\nu$,

$$\bar{K}(s,t,x) = \begin{cases} 0, & (s,t,x) \in Z_{m'} \times \Gamma \times \mathbb{R}^\nu, \\ K_\delta(s,t,x), & (s,t,x) \in (\Delta \backslash Z_{m'}) \times \Gamma \times \mathbb{R}^\nu. \end{cases}$$

The map $\bar{K}$ satisfies (2.7), (3.2.1), and (3.2.3) just as $K_\delta$ does. Moreover $\bar{K}$ satisfies condition (3.2.2) with the function $\varphi: \Delta^a \to \mathbb{R}_+$,

$$\varphi(s) = \begin{cases} p(s)\, \alpha(s) + c(s), & s \in Z_{m'} \\ \varepsilon/2a^\lambda, & s \in S_{m'}. \end{cases} \quad \blacksquare$$

## 4. Proofs of Theorems

(4.1) *Proof of Theorem* (2.2). We check that the maps $f = I - F$ and $f_n = I - F_n$ $(n \in \mathbb{N})$ satisfy the assumptions of lemma (3.1). The map $f$ is a compact vector field; hence it satisfies the Palais–Smale condition (3.1.1). Moreover $f$ is a closed map; hence $(2\varepsilon_n \bar{U}_n) \cap f(\partial U) = \emptyset$ for sufficiently large $n \in \mathbb{N}$. Then, for every $y \in \varepsilon_n \bar{U}_n$, the homotopy

$$H_{n,y}\colon [0,1] \times \bar{U} \to E, \qquad H_{n,y}(t,x) = (1-t)\,F(x) + t(F_n(x) - y)$$

is compact and does not have fixed points on $\partial U$. Thus, by (1.4), for every $y \in \varepsilon_n \bar{U}_n$ $\deg(f_n, U, y) = \deg(f, U, 0) \neq 0$, and by (1.3), $y \in f_n(U)$. By virtue of (2.2.2), $\tilde{f}_n\colon f_n^{-1}(\varepsilon_n \bar{U}_n) \to \varepsilon_n \bar{U}_n$ is a one-to-one compact vector field; hence it is a homeomorphism.

The thesis follows now from Lemma (3.1). ∎

(4.2) *Proof of Theorem* (2.3). The compact homotopy

$$H\colon [0,1] \times E \to E, \qquad H(t,x) = tF(x),$$

connects $F$ with the zero map. Hence $\deg(I-F, E, 0) = \deg(I, E, 0) = 1$, and the assumptions of Theorem (4.1) are fullfilled. ∎

(4.3) *Proof of Theorem* (2.5). Let $f = I - F$. From (2.4.1) and (2.4.2) it follows that the map $f|_{(x+(1/2)\bar{\mathcal{O}})}$ is injective for every $x \in E$. Moreover if $f(x_1) = f(x_2)$ then $(x_1 + \frac{1}{2}\bar{\mathcal{O}}) \cap (x_2 + \frac{1}{2}\bar{\mathcal{O}}) = \emptyset$.

We show that there exists a neighbourhood of zero $V$ such that $(f(x) + V) \cap f(x + \frac{1}{2}\partial\mathcal{O}) = \emptyset$ for every $x \in E$. Suppose, on the contrary, that there exist sequences $\{x_n\}, \{y_n\} \subseteq E$ such that $y_n \in x_n + \frac{1}{2}\partial\mathcal{O}$ $(n \in \mathbb{N})$ and $f(x_n) - f(y_n) \to 0$. For every $n \in \mathbb{N}$ we have $x_n - y_n \in \mathcal{O}$, so that $F(x_n) - F(y_n) \in H(x_n - y_n)$. By the compactness of $H$ we can assume that $F(x_n) - F(y_n) \to z$; hence $x_n - y_n \to z$ and $z \in \frac{1}{2}\partial\mathcal{O} \subseteq \mathcal{O}$. Since the graph of $H$ is closed $z \in H(z)$, $z = 0$, which contradicts that $z \in \frac{1}{2}\,\partial\mathcal{O}$. Let us note that we can assume $V$ to be convex and symmetric.

For every $x \in E$ the map $F|_{(x+1/2)\bar{\mathcal{O}})}$ is compact since $F(x + \frac{1}{2}\bar{\mathcal{O}}) \subseteq F(x) + H(\bar{\mathcal{O}})$. Thus $f|_{(x+(1/2)\bar{\mathcal{O}})}$ is an injective compact vector field and by the Nagumo theorem (1.6) $\deg(f, x + \frac{1}{2}\mathcal{O}, f(x)) \neq 0$. Since $f(x) + V$ is a connected set and $(f(x) + V) \cap f(x + \frac{1}{2}\,\partial\mathcal{O}) = \emptyset$ we have, by (1.5), $\deg(f, x + \frac{1}{2}\mathcal{O}, y) = \deg(f, x + \frac{1}{2}\mathcal{O}, f(x)) \neq 0$ for every $y \in f(x) + V$. Hence, by (1.3), $f(x) + V \subseteq f(x + \frac{1}{2}\mathcal{O})$ for every $x \in E$.

It easily follows from the above inclusion that $f(E)$ is open and $f(E) = E$. Moreover the map $f\colon E \to E$ is a covering, since for every $y \in E$, $y + V$ is a well covered neighbourhood. Actually, $y + V = \bigcup \{\mathcal{O}_x \mid x \in f^{-1}(y)\}$ where the sets $\mathcal{O}_x = f^{-1}(y+V) \cap (x + \frac{1}{2}\mathcal{O})$, $x \in f^{-1}(y)$ are disjoint one with another and are mapped by $f$ homeomorphically onto $y + V$. The space $E$ is simply connected; thus the covering $f$ is a homeomorphism. ∎

(4.4) *Proof of Theorem* (2.6). Theorem (2.6) follows now from Theorem (2.5) and Lemma (3.1). ∎

(4.5) *Proof of Theorem* (2.8). The set of solutions of the problem (D) coincides with the set of solutions of the equation

$$u(x, y) = h(x) + g(y) - h(0) + \int_0^x \int_0^y K(\xi, \eta, u(\xi, \eta))\, d\xi\, d\eta.$$

We show that for arbitrary locally integrable functions $p, c\colon \Delta \to \mathbb{R}_+$ there exists a continuous function $\alpha\colon \Delta \to \mathbb{R}_+$ such that for every solution $u\colon \Delta \to \mathbb{R}^\nu$ of the problem (D)

$$|u(x, y)| \leqslant \alpha(x, y), \qquad (x, y) \in \Delta,$$

provided $K$ satisfies the condition (2.7.3) with the functions $p$ and $c$. Actually, if $u\colon \Delta \to \mathbb{R}^\nu$ is a solution of (D) then

$$|u(x, y)| \leqslant |h(x)| + |g(y)| + |h(0)| + \int_0^x \int_0^y c(\xi, \eta)\, d\xi\, d\eta$$

$$+ \int_0^x \int_0^y p(\xi, \eta)|u(\xi, \eta)|\, d\xi\, d\eta, \qquad (x, y) \in \Delta.$$

Let $\tilde{h}(x) = \sup\{|h(\xi)| \mid \xi \in [0, x]\}$, $\tilde{g}(y) = \sup\{|g(\eta)| \mid \eta \in [0, y]\}$, and $\tilde{c}(x, y) = \tilde{h}(0) + \tilde{h}(x) + \tilde{g}(y) + \int_0^x \int_0^y c(\xi, \eta)\, d\xi, d\eta$. Then

$$|u(x, y)| \leqslant \tilde{c}(x, y) + \int_0^x \int_0^y p(\xi, \eta)|u(\xi, \eta)|\, d\xi\, d\eta, \qquad (x, y) \in \Delta,$$

where $\tilde{c}\colon \Delta \to \mathbb{R}_+$ is a continuous nondecreasing function (i.e., if $0 \leqslant x \leqslant x'$ and $0 \leqslant y \leqslant y'$ then $\tilde{c}(x, y) \leqslant \tilde{c}(x', y')$. Hence, by the Wendroff inequality [2],

$$|u(x, y)| \leqslant \tilde{c}(x, y) \cdot \exp\left(\int_0^x \int_0^y p(\xi, \eta)|u(\xi, \eta)|\, d\xi\, d\eta\right), \qquad (x, y) \in \Delta,$$

and it suffices to put $\alpha(x, y)$ equal to the right side of the above inequality.

Let $\psi\colon \mathbb{R}^\nu \to [0, 1]$ be a continuous function such that $\psi(u) = 1$ for $u \in \bar{B}_\nu(1)$ and $\psi(u) = 0$ for $u \notin \bar{B}_\nu(2)$. Let us define

$$\tilde{K}\colon \Delta \times \mathbb{R}^\nu \to \mathbb{R}^\nu, \qquad \tilde{K}(x, y, u) = \psi\left(\frac{u}{\alpha(x, y) + 1}\right) \cdot K(x, y, u).$$

The map $\tilde{K}$ satisfies (2.7) with the same functions $p$ and $c$ as $K$, and

$\tilde{K}(x, y, u) = K(x, y, u)$, when $|u| \leqslant \alpha(x, y)$. Hence the set of solutions of (D) coincides with the set of solutions of the problem

$$\begin{aligned}
&u_{xy}(x, y) = \tilde{K}(x, y, u(x, y)), \qquad \text{for a.a. } (x, y) \in \Delta,\\
&u(0, y) = g(y),\ u(x, 0) = h(x), \qquad \text{for } x, y \in \mathbb{R}_+ \qquad\qquad (\tilde{\mathrm{D}})\\
&u\colon \Delta \to \mathbb{R}^\nu \text{ is an absolutely continuous map.}
\end{aligned}$$

Moreover $\tilde{K}$ satisfies the condition (3.2.1): $\operatorname{supp} \tilde{K} \subseteq \Omega_{\tilde{\alpha}}$, where $\tilde{\alpha}(s) = 2\alpha(s) + 2$.

Let $n \in \mathbb{N}$ be arbitrarily chosen and let $\varepsilon = 1/n$, $a = n$. By virtue of Lemma (3.2) there exists a map $K_n\colon \Delta \times \mathbb{R}^\nu \to \mathbb{R}^\nu$ satisfying conditions (2.7), (3.2.1), (3.2.2), and (3.2.3) (with appropriate functions $p_n, c_n, \alpha_n, L_n\colon \Delta \to \mathbb{R}_+$, and $\varphi_n\colon \Delta^n \to \mathbb{R}_+$).

We define the following maps

$$F\colon \mathscr{C}(\Delta, \mathbb{R}^\nu) \to \mathscr{C}(\Delta, \mathbb{R}^\nu),$$

$$F(u)(x, y) = h(x) + g(y) - h(0) + \int_0^x \int_0^y \tilde{K}(\xi, \eta, u(\xi, \eta))\, d\xi\, d\eta$$

and

$$F_n\colon \mathscr{C}(\Delta, \mathbb{R}^\nu) \to \mathscr{C}(\Delta, \mathbb{R}^\nu),$$

$$F_n(u)(x, y) = h(x) + g(y) - h(0) + \int_0^x \int_0^y K_n(\xi, \eta, u(\xi, \eta))\, d\xi\, d\eta,$$

$$(n \in \mathbb{N}).$$

The maps $F$ and $F_n$ $(n \in \mathbb{N})$ are continuous. Actually, it suffices to show that $F(u_k) \to F(u)$, when $u_k \to u$ in $\mathscr{C}(\Delta, \mathbb{R}^\nu)$. Let $\varepsilon > 0$ and $n \in \mathbb{N}$ be arbitrary and let us define

$$\begin{aligned}
S_m = \{&(x, y) \in \Delta^n \mid \forall_{u, u' \in \mathbb{R}^\nu}\ |u - u'| < 1/m\\
&\Rightarrow |\tilde{K}(x, y, u) - \tilde{K}(x, y, u')| \leqslant \varepsilon/2n^2\} \qquad (m \in \mathbb{N}).
\end{aligned}$$

$\{S_m\}$ is a nondecreasing sequence of measurable sets and $\bigcup_{m=1}^\infty S_m = \Delta^n$; hence there exists $m' \in \mathbb{N}$ such that

$$\int_{\Delta^n \setminus S_{m'}} (p(\xi, \eta)\, \tilde{\alpha}(\xi, \eta) + c(\xi, \eta))\, d\xi\, d\eta < \varepsilon/4.$$

Let us take $k' \in \mathbb{N}$ such that $q_n(u_k - u) < 1/m'$ for all $k > k'$. Then for all $k > k'$ and $(x, y) \in \Delta^n$

$$
\begin{aligned}
&|(F(u_k) - F(u))(x, y)| \\
&\quad \leqslant \int_0^x \int_0^y |\tilde{K}(\xi, \eta, u_k(\xi, \eta)) - \tilde{K}(\xi, \eta, u(\xi, \eta))| \, d\xi \, d\eta \\
&\quad \leqslant \int_{S_{m'}} (\varepsilon/2n^2) \, d\xi \, d\eta \\
&\qquad + \int_{\Delta^n \setminus S_{m'}} 2(p(\xi, \eta) \, \tilde{\alpha}(\xi, \eta) + c(\xi, \eta)) \, d\xi \, d\eta < \varepsilon;
\end{aligned}
$$

hence $q_n(F(u_k) - F(u)) < \varepsilon$ for $k < k'$. Thus $q_n(F(u_k) - F(u)) \to 0$ when $k \to +\infty$ for arbitrary $n \in \mathbb{N}$.

By the inequality $|\tilde{K}(x, y, u(x, y))| \leqslant p(x, y) \, \tilde{\alpha}(x, y) + c(x, y)$ for every $u \in \mathscr{C}(\Delta, \mathbb{R}^\nu)$, $F(\mathscr{C}(\Delta, \mathbb{R}^\nu))$ is a bounded set of equally continuous maps; hence the map $F$ is compact. Further it is easily seen that $\{F_n\}$ is a $\{1/n\}$-approximation of $F$.

We show that for every $n \in \mathbb{N}$ the map $F_n$ satisfies the Lasota–Opial condition with the set-valued map

$$
\begin{aligned}
&H_n : \mathscr{C}(\Delta, \mathbb{R}^\nu) \to \varkappa(\mathscr{C}(\Delta, \mathbb{R}^\nu)), \\
&H_n(u) = \left\{ \int_0^x \int_0^y \lambda(\xi, \eta) \, d\xi \, d\eta \;\middle|\; \lambda : \Delta \to \mathbb{R}^\nu \text{ is measurable}, \right. \\
&\qquad |\lambda(x, y)| \leqslant L_n(x, y) |u(x, y)| \text{ and} \\
&\qquad \left. |\lambda(x, y)| \leqslant 2(p_n(x, y) \, \alpha_n(x, y) + c_n(x, y)) \text{ for a.a. } (x, y) \in \Delta \right\}.
\end{aligned}
$$

The map $H_n$ is compact [7, Propositions (3.8) and (3.5)]. If $u \in H_n(u)$ then

$$
|u(x, y)| \leqslant \int_0^x \int_0^y L_n(\xi, \eta) |u(\xi, \eta)| \, d\xi \, d\eta, \qquad (x, y) \in \Delta,
$$

and by the Wendroff inequality [2] $u(x, y) = 0$ for all $(x, y) \in \Delta$. Moreover for all $u, v \in \mathscr{C}(\Delta, \mathbb{R}^\nu)$

$$
\begin{aligned}
&|(F_n(u) - F_n(v))(x, y)| \\
&\quad \leqslant \int_0^x \int_0^y |K_n(\xi, \eta, u(\xi, \eta)) - K_n(\xi, \eta, v(\xi, \eta))| \, d\xi \, d\eta,
\end{aligned}
$$

where

$$|K_n(\xi, \eta, u(\xi, \eta)) - K_n(\xi, \eta, v(\xi, \eta))| \leqslant L_n(\xi, \eta)|u(\xi, \eta) - v(\xi, \eta)|$$

and

$$|K_n(\xi, \eta, u(\xi, \eta)) - K_n(\xi, \eta, v(\xi, \eta))| \leqslant 2\,(p_n(\xi, \eta)\,\alpha_n(\xi, \eta) + c_n(\xi, \eta))$$

for $(\xi, \eta) \in \Delta$, and so $F_n(u) - F_n(v) \in H_n(u - v)$.

By Theorem (2.6) Fix $F$ is an $\mathscr{R}_\delta$-set in $\mathscr{C}(\Delta, \mathbb{R}^\nu)$. Since Fix $F$ coincides with the set of solutions of (D), the proof is finished. ∎

(4.6) *Proof of Theorem* (2.9). Let $x: \mathbb{R}_+ \to \mathbb{R}^\nu$ be a continuous solution of Eq. (I). Then

$$|x(t)| \leqslant \int_0^t |K(s, t, x(s))|\, ds \leqslant \int_0^t c(s)\, ds + \int_0^t p(s)|x(s)|\, ds$$

for all $t \in \mathbb{R}_+$ and by the Gronwall inequality $|x(t)| \leqslant \alpha(t)$ for all $t \in \mathbb{R}_+$, where $\alpha: \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function given by

$$\alpha(t) = \left(\int_0^t c(s)\, ds\right) \cdot \exp\left(\int_0^t p(s)\, ds\right).$$

Let $\psi: \mathbb{R}^\nu \to [0, 1]$ be a continuous function such that $\psi(x) = 1$ for $x \in \bar{B}_\nu(1)$ and $\psi(x) = 0$ for $x \notin \bar{B}_\nu(2)$. Let us define

$$\tilde{K}: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\nu \to \mathbb{R}^\nu, \qquad \tilde{K}(s, t, x) = \psi\left(\frac{x}{\alpha(s) + 1}\right) \cdot K(s, t, x).$$

The map $\tilde{K}$ satisfies (2.7) with the same functions $p$ and $c$ as $K$, and $\tilde{K}(s, t, x) = K(s, t, x)$, when $|x| \leqslant \alpha(s)$. Hence the set of continuous solutions of the equation

$$x(t) = \int_0^t \tilde{K}(s, t, x(s))\, ds \tag{$\tilde{\mathrm{I}}$}$$

coincides with the set of continuous solutions of (I). Moreover $\tilde{K}$ satisfies (3.2.1): sup $\tilde{K} \subseteq \Omega_{\tilde{\alpha}}$, where $\tilde{\alpha}(s) = 2\alpha(s) + 2$.

Let $n \in \mathbb{N}$ be arbitrarily chosen and let $\varepsilon = 1/n$, $a = n$. By virtue of Lemma (3.2) there exists a map $K_n: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\nu \to \mathbb{R}^\nu$ satisfying the conditions (2.7), (3.2.1), (3.2.2), and (3.2.3) (with appropriate functions $p_n, c_n, \alpha_n, L_n: \mathbb{R}_+ \to \mathbb{R}_+$, and $\varphi_n: [0, n] \to \mathbb{R}_+$).

We show that the following formulae

$$F(x)(t) = \int_0^t \tilde{K}(s, t, x(s))\, ds$$

and

$$F_n(x)(t) = \int_n^t K_n(s, t, x(s))\, ds \qquad (n \in \mathbb{N})$$

define compact maps $F, F_n: \mathscr{C}(\mathbb{R}_+, \mathbb{R}^\nu) \to \mathscr{C}(\mathbb{R}_+, \mathbb{R}^\nu)$ $(n \in \mathbb{N})$.

First we show that $F(\mathscr{C}(\mathbb{R}_+, \mathbb{R}^\nu))$ is a set of equally continuous maps. Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. There exists $\delta' > 0$ such that

$$\int_Z (p(s)\, \tilde{\alpha}(s) + c(s))\, ds < \varepsilon/4$$

for every measurable $Z \in [0, n]$ provided $|Z| < \delta'$. If

$$S_m = \{s \in [0, n] \mid \forall_{z \in \mathbb{R}^\nu} \forall_{t, t' \in [0, n]}\ |t - t'| < 1/m$$
$$\Rightarrow |\tilde{K}(s, t, z) - \tilde{K}(s, t', z)| \leqslant \varepsilon/4n\} \qquad (m \in \mathbb{N})$$

then there exists $m' \in \mathbb{N}$ such that $|[0, n] \setminus S_{m'}| < \delta'$. Let us take $\delta = \min(\delta', 1/m')$. Then for all $t$, $t' \in [0, n]$ such that $0 < t - t' < \delta$ and $x \in \mathscr{C}(\mathbb{R}_+, \mathbb{R}^\nu)$ we have

$$\begin{aligned}
&|F(x)(t) - F(x)(t')| \\
&\quad \leqslant \int_0^t |\tilde{K}(s, t, x(s)) - \tilde{K}(s, t', x(s))|\, ds + \int_{t'}^t |\tilde{K}(s, t', x(s))|\, ds \\
&\quad \leqslant \int_{S_{m'}} (\varepsilon/4n)\, ds + \int_{[0, n] \setminus S_{m'}} 2(p(s)\, \tilde{\alpha}(s) + c(s))\, ds \\
&\qquad + \int_{t'}^t (p(s)\, \tilde{\alpha}(s) + c(s))\, ds < \varepsilon.
\end{aligned}$$

We have shown that $F(\mathscr{C}(\mathbb{R}_+, \mathbb{R}^\nu))$ is a set of equally continuous maps and in particular $F(\mathscr{C}(\mathbb{R}_+, \mathbb{R}^\nu)) \in \mathscr{C}(\mathbb{R}_+, \mathbb{R}^\nu)$.

Since

$$|\tilde{K}(s, t, x(s))| \leqslant p(s)\, \tilde{\alpha}(s) + c(s),$$

for all $(s, t) \in \mathbb{R}_+ \times \mathbb{R}_+$ and $x \in \mathscr{C}(\mathbb{R}_+, \mathbb{R}^\nu)$, the set $F(\mathscr{C}(\mathbb{R}_+, \mathbb{R}^\nu))$ is bounded in $\mathscr{C}(\mathbb{R}_+, \mathbb{R}^\nu)$; hence it is relatively compact. One can establish the continuity of $F$ just as in Proof (4.5). Thus we have shown compactness of $F$, and the proof for $F_n$ is the same.

It is easily seen that $\{F_n\}$ is a $\{1/n\}$-approximation of $F$. Let us note that for every $n \in \mathbb{N}$ and $y \in \mathscr{C}(\mathbb{R}_+, \mathbb{R}^\nu)$ the equation $x - F_n(x) = y$ has at most one solution. Actually, if $x_1 - F_n(x_1) = x_2 - F_n(x_2)$ then

$$|x_1(t) - x_2(t)| \leqslant \int_0^t |K_n(s, t, x_1(s)) - K_n(s, t, x_2(s))| \, ds$$

$$\leqslant \int_0^t L_n(s) |x_1(s) - x_2(s)| \, ds$$

and by the Gronwall inequality $x_1(t) - x_2(t) = 0$ for all $t \in \mathbb{R}_+$.

We have shown that the map $F$ satisfies the assumptions of Theorem (2.3); hence Fix $F$ is an $\mathscr{R}_\delta$-set in $\mathscr{C}(\mathbb{R}_+, \mathbb{R}^\nu)$ and so is the set of solutions of Eq. (I). ∎

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